

QUASI-ISOMETRIC CO-HOPFICITY OF NON-UNIFORM LATTICES IN RANK-ONE SEMI-SIMPLE LIE GROUPS

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ABSTRACT. We prove that if G is a non-uniform lattice in a rank-one semi-simple Lie group $\neq \text{Isom}(\mathbb{H}_{\mathbb{R}}^2)$ then G is quasi-isometrically co-Hopf. This means that every quasi-isometric embedding $G \rightarrow G$ is coarsely surjective and thus is a quasi-isometry.

1. INTRODUCTION

The notion of co-Hopfity plays an important role in group theory. Recall that a group G is said to be *co-Hopf* if G is not isomorphic to a proper subgroup of itself, that is, if every injective homomorphism $G \rightarrow G$ is surjective. A group G is *almost co-Hopf* if for every injective homomorphism $\phi : G \rightarrow G$ we have $[G : \phi(G)] < \infty$. Clearly, being co-Hopf implies being almost co-Hopf. The converse is not true: for example, for any $n \geq 1$ the free abelian group \mathbb{Z}^n is almost co-Hopf but not co-Hopf.

It is easy to see that any freely decomposable group is not co-Hopf. In particular, a free group of rank at least 2 is not co-Hopf. It is also well-known that finitely generated nilpotent groups are always almost co-Hopf and, under some additional restrictions, also co-Hopf [1]. An important result of Sela [17] states that a torsion-free non-elementary word-hyperbolic group G is co-Hopf if and only if G is freely indecomposable. Partial generalizations of this result are known for certain classes of relatively hyperbolic groups, by the work of Belegradek and Szczepański [2]. Co-Hopfity has also been extensively studied for 3-manifold groups and for Kleinian groups. Delzant and Potyagailo [9] gave a complete characterization of co-Hopfian groups among non-elementary geometrically finite Kleinian groups without 2-torsion.

A counterpart algebraic notion is that of Hopfity. A group G is said to be *Hopfian* if every surjective endomorphism $G \rightarrow G$ is necessarily injective, and hence is an automorphism of G . This notion is also extensively studied in geometric group theory. In particular, an important result of Sela [18] shows that every torsion-free word-hyperbolic group is Hopfian. The notion of Hopfity admits a number of interesting “virtual” variations. Thus a group G is called *cofinitely Hopfian* if every endomorphism of G whose image is of finite index in G , is an automorphism of G , see, for example [7].

A key general theme in geometric group theory is the study of “large-scale” geometric properties of finitely generated groups. Recall that if (X, d_X) and (Y, d_Y) are metric spaces, a map $f : X \rightarrow Y$ is called a *coarse embedding* if there exist

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monotone non-decreasing functions $\alpha, \omega : [0, \infty) \rightarrow \mathbb{R}$ such that $\alpha(t) \leq \omega(t)$, that $\lim_{t \rightarrow \infty} \alpha(t) = \infty$, and such that for all $x, x' \in X$ we have

$$(*) \quad \alpha(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \omega(d_X(x, x')).$$

If d_X is a path metric, then for any coarse embedding $f : X \rightarrow Y$ the function $\omega(t)$ can be chosen to be affine, that is, of the form $\omega(t) = at + b$ for some $a, b \geq 0$.

A coarse map f is called a *coarse equivalence* if f is *coarsely surjective*, that is, if there is $C \geq 0$ such that for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq C$. A map $f : X \rightarrow Y$ is called a *quasi-isometric embedding* if f is a coarse embedding and the functions $\alpha(t), \omega(t)$ in $(*)$ can be chosen to be affine, that is, of the form $\alpha(t) = \frac{1}{\lambda}t - \epsilon$, $\omega(t) = \lambda t + \epsilon$ where $\lambda \geq 1$, $\epsilon \geq 0$. Finally, a map $f : X \rightarrow Y$ is a *quasi-isometry* if f is a quasi-isometric embedding and f is coarsely surjective.

The notion of co-Hopfity has the following natural counterpart for metric spaces. We say that a metric space X is *quasi-isometrically co-Hopf* if every quasi-isometric embedding $X \rightarrow X$ is coarsely surjective, that is, if every quasi-isometric embedding $X \rightarrow X$ is a quasi-isometry. More generally, a metric space X is called *coarsely co-Hopf* if every coarse embedding $X \rightarrow X$ is coarsely surjective. Clearly, if X is coarsely co-Hopf then X is quasi-isometrically co-Hopf. If G is a finitely generated group with a word metric d_G corresponding to some finite generating set of G , then every injective homomorphism $G \rightarrow G$ is a coarse embedding. This easily implies that if (G, d_G) is coarsely co-Hopf then the group G is almost co-Hopf.

Example 1.1. The real line \mathbb{R} is coarsely co-Hopf (and hence quasi-isometrically co-Hopf). This follows from the fact that any coarse embedding must send the ends of \mathbb{R} to distinct ends. Since \mathbb{R} has two ends, a coarse embedding induces a bijection on the set of ends of \mathbb{R} . It is then not hard to see that a coarse embedding from \mathbb{R} to \mathbb{R} must be coarsely surjective. See [6] for the formal definition of ends of a metric space.

Example 1.2. The rooted regular binary tree T_2 is not quasi-isometrically co-Hopf. We can identify the set of vertices of T_2 with the set of all finite binary sequences. The root of T_2 is the empty binary sequence ϵ and for a finite binary sequence x its left child is the sequence $0x$ and the right child is the sequence $1x$. Consider the map $f : T_2 \rightarrow T_2$ which maps T_2 isometrically to a copy of itself that “hangs below” the vertex 0. Thus $f(x) = 0x$ for every finite binary sequence x . Then f is an isometric embedding but the image $f(T_2)$ is not co-bounded in T_2 since it misses the entire infinite branch located below the vertex 1.

Example 1.3. Consider the free group $F_2 = F(a, b)$ on two generators. Then F_2 is not quasi-isometrically co-Hopf.

The Cayley graph X of F_2 is a regular 4-valent tree with every edge of length 1. We may view X in the plane so that every vertex has one edge directed upward, and three downward. Picking a vertex v_0 of X , denote its left branch by X_1 and the remainder of the tree by X_2 . We have $X_1 \cup X_2 = X$, and X_1 is a rooted ternary tree. Define a quasi-isometric embedding $f : X \rightarrow X$ by taking f to be a shift on X_1 (defined similarly to Example 1.2) and the identity on X_2 . The map f is not coarsely surjective, but it is a quasi-isometric embedding. Moreover, for any vertices x, x' of X we have $|d(f(x), f(x')) - d(x, x')| \leq 1$.

One can also see that $F_2 = F(a, b)$ is not quasi-isometrically co-Hopf for algebraic reasons. Let $u, v \in F(a, b)$ with $[u, v] \neq 1$. Then there is an injective homomorphism

$h : F(a, b) \rightarrow F(a, b)$ such that $h(a) = u$ and $h(b) = v$. This homomorphism f is always a quasi-isometric embedding of $F(a, b)$ into itself.

If, in addition, u and v are chosen so that $\langle u, v \rangle \neq F(a, b)$ then $[F(a, b) : h(F(a, b))] = \infty$ and the image $h(F(a, b))$ is not co-bounded in $F(a, b)$.

Thus, the group F_2 is not almost co-Hopf and not quasi-isometrically co-Hopf.

Example 1.4. There do exist finitely generated groups that are algebraically co-Hopf but not quasi-isometrically co-Hopf. The simplest example of this kind is the solvable Baumslag-Solitar group $B(1, 2) = \langle a, t | t^{-1}at = a^2 \rangle$. It is well-known that $B(1, 2)$ is co-Hopf.

To see that $B(1, 2)$ is not quasi-isometrically co-Hopf we use the fact that $B(1, 2)$ admits an isometric properly discontinuous co-compact action on a proper geodesic metric space X that is “foliated” by copies of the hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$. We refer the reader to the paper of Farb and Mosher [12] for a detailed description of the space X , and will only briefly recall the properties of X here.

Topologically, X is homeomorphic to the product $\mathbb{R} \times T_3$ where T_3 is an infinite 3-regular tree (drawn upwards): there is a natural projection $p : X \rightarrow T_3$ whose fibers are homeomorphic to \mathbb{R} . The boundary of T_3 is decomposed into two sets: the “lower boundary” consisting of a single point u and the “upper boundary” $\partial_{\delta} X$ which is homeomorphic to the Cantor set (and can be identified with the set of dyadic rationals). For any bi-infinite geodesic ℓ in T_3 from u to a point of $\partial_{\delta} X$ the full- p -preimage of ℓ in X is a copy of the hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$ (in the upper-half plane model). The p -preimage of any vertex of T_3 is a horizontal horocycle in the $\mathbb{H}_{\mathbb{R}}^2$ -“fibers”. Any two \mathbb{H}^2 -fibers intersect along a complement of a horoball in $\mathbb{H}_{\mathbb{R}}^2$.

Similar to the above example for $F(a, b)$, we can take a quasi-isometric embedding $f : T_3 \rightarrow T_3$ whose image misses an infinite subtree in T_3 and such that $|d(x, x') - d(f(x), f(x'))| \leq 1$ for any vertices x, x' of T_3 . It is not hard to see that this map f can be extended along the p -fibers to a map $\tilde{f} : X \rightarrow X$ such that \tilde{f} is a quasi-isometric embedding but not coarsely surjective. Since X is quasi-isometric to $B(1, 2)$, it follows that $B(1, 2)$ is not quasi-isometrically co-Hopf.

Example 1.5. Grigorchuk’s group G of intermediate growth provides another interesting example of a group that is not quasi-isometrically co-Hopf. This group G is finitely generated and can be realized as a group of automorphisms of the regular binary rooted tree T_2 . The group G has a number of unusual algebraic properties: it is an infinite 2-torsion group, it has intermediate growth, it is amenable but not elementary amenable and so on. See Ch. VIII in [8] for detailed background on the Grigorchuk group. It is known that there exists a subgroup K of index 16 in G such that $K \times K$ is isomorphic to a subgroup of index 64 in G . The map $K \rightarrow K \times K, k \mapsto (k, 1)$ is clearly a quasi-isometric embedding which is not coarsely surjective. Since both K and $K \times K$ are quasi-isometric to G , it follows that G is not quasi-isometrically co-Hopf.

For Gromov-hyperbolic groups and spaces quasi-isometric co-Hopfity is closely related to the properties of their hyperbolic boundaries. We say that a compact metric space K is *topologically co-Hopf* if K is not homeomorphic to a proper subset of itself. We say that K is *quasi-symmetrically co-Hopf* if every quasi-symmetric map $K \rightarrow K$ is surjective. Note that for a compact metric space K being topologically co-Hopf obviously implies being quasi-symmetrically co-Hopf.

Example 1.6. A recent important result of Merenkov [15] shows that the converse implication does not hold. He constructed a round Sierpinski carpet \mathbb{S} such that \mathbb{S} is quasi-symmetrically co-Hopf. Since \mathbb{S} is homeomorphic to the standard “square” Sierpinski carpet, clearly \mathbb{S} is not topologically co-Hopf.

It is well-known (see, for example, [3]) that if X, Y are proper Gromov-hyperbolic geodesic metric spaces, then any quasi-isometric embedding $f : X \rightarrow Y$ induces a quasi-symmetric topological embedding $\partial f : \partial X \rightarrow \partial Y$ between their hyperbolic boundaries. It is then not hard to see that if G is a word-hyperbolic group whose hyperbolic boundary ∂G is quasi-symmetrically co-Hopf (e.g. if it is topologically co-Hopf), then G is quasi-isometrically co-Hopf. This applies, for example, to any word-hyperbolic groups whose boundary ∂G is homeomorphic to an n -sphere (with $n \geq 1$), such as fundamental groups of closed Riemannian manifolds with all sectional curvatures ≤ -1 .

The main result of this paper is the following:

Theorem 1.7. *Let G be a non-uniform lattice in a rank-one semi-simple real Lie group other than $\text{Isom}(\mathbb{H}_{\mathbb{R}}^2)$. Then G is quasi-isometrically co-Hopf.*

Thus, for example, if M is a complete finite volume non-compact hyperbolic manifold of dimension $n \geq 3$ then $\pi_1(M)$ is quasi-isometrically co-Hopf. Note that if G is a non-uniform lattice in $\text{Isom}(\mathbb{H}_{\mathbb{R}}^2)$ then the conclusion of Theorem 1.7 does not hold since G is a virtually free group.

If G is a uniform lattice in a rank-one semi-simple real Lie group (including possibly a lattice in $\text{Isom}(\mathbb{H}_{\mathbb{R}}^2)$) then G is Gromov-hyperbolic with the boundary ∂G being homeomorphic to \mathbb{S}^n (for some $n \geq 1$). In this case it is easy to see that G is also quasi-isometrically co-Hopf since every topological embedding from \mathbb{S}^n to itself is necessarily surjective.

Convention 1.8. *From now on and for the remainder of this paper let $X \neq \mathbb{H}_{\mathbb{R}}^2$ be a rank-one negatively curved symmetric space with metric d_X (or just d in most cases). Namely, X is isometric to a hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$ (with $n \geq 3$), $\mathbb{H}_{\mathbb{C}}^n$ (with $n \geq 2$), $\mathbb{H}_{\mathbb{H}}^n$ over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}_{\mathbb{O}}^2$.*

If G is as in Theorem 1.7, then G acts properly discontinuously (but with a non-compact quotient) by isometries on such a space X and there exists a G -invariant collection \mathcal{B} of disjoint horoballs in X such that $(X \setminus \mathcal{B})/G$ is compact. The “truncated” space $\Omega = X \setminus \mathcal{B}$, endowed with the induced path-metric d_{Ω} is quasi-isometric to the group G by the Milnor-Schwartz Lemma. Thus it suffices to prove that (Ω, d_{Ω}) is quasi-isometrically co-Hopf.

Richard Schwartz [16] established quasi-isometric rigidity for non-uniform lattices in rank-one semi-simple Lie groups and we use his proof as a starting point.

First, using coarse cohomological methods (particularly techniques of Kapovich-Kleiner [14]), we prove that spaces homeomorphic to \mathbb{R}^n with “reasonably nice” metrics are coarsely co-Hopf. This result applies to the Euclidean space \mathbb{R}^n itself, to simply connected nilpotent Lie groups, to the rank-one symmetric spaces X mentioned above, as well as to the horospheres in X . Let $f : (\Omega, d_{\Omega}) \rightarrow (\Omega, d_{\Omega})$ be a quasi-isometric embedding. Schwartz’ work implies that for every peripheral horosphere σ in Ω there exists a unique peripheral horosphere σ' of X such that $f(\sigma)$ is contained in a bounded neighborhood of σ' . Using coarse co-Hopfity of

horospheres, mentioned above, we conclude that f gives a quasi-isometry (with controlled constants) between σ and σ' . Then, following Schwartz, we extend the map f through each peripheral horosphere to the corresponding peripheral horoball B in X . We then argue that the extended map $\hat{f} : X \rightarrow X$ is a coarse embedding. Using coarse co-Hopfity of X , it follows that \hat{f} is coarsely surjective, which implies that the original map $f : (\Omega, d_\Omega) \rightarrow (\Omega, d_\Omega)$ is coarsely surjective as well.

It seems likely that the proof of Theorem 1.7 generalizes to an appropriate subclass of relatively hyperbolic groups. However, a more intriguing question is to understand what happens for higher-rank lattices:

Problem 1.9. Let G be a non-uniform lattice in a semi-simple real Lie group of rank ≥ 2 . Is G quasi-isometrically co-Hopf?

Unlike the groups considered in the present paper, higher-rank lattices are not relatively hyperbolic. Quasi-isometric rigidity for higher-rank lattices is known to hold, by the result of Eskin [11], but the proofs there are quite different from the proof of Schwartz in the rank-one case.

Another natural question is:

Problem 1.10. Let G be as in Theorem 1.7. Is G coarsely co-Hopf?

Our proof only yields quasi-isometric co-Hopfity, and it is possible that coarse co-Hopfity actually fails in this context.

The result of Merenkov (Example 1.6) produces the first example of a compact metric space K which is quasi-symmetrically co-Hopf but not topologically co-Hopf. Topologically, K is homeomorphic to the standard Sierpinski carpet and there exists a word-hyperbolic group (in fact a Kleinian group) with boundary homeomorphic to K . However, the metric structure on the Sierpinski carpet in Merenkov's example is not "group-like" and is not quasi-symmetric to the visual metric on the boundary of a word-hyperbolic group.

Problem 1.11. Does there exist a word-hyperbolic group G such that ∂G (with the visual metric) is quasi-symmetrically co-Hopf (and hence G is quasi-isometrically co-Hopf), but such that ∂G is not topologically co-Hopf? In particular, do there exist examples of this kind where ∂G is homeomorphic to the Sierpinski carpet or the Menger curve?

The above question is particularly interesting for the family of hyperbolic buildings $I_{p,q}$ constructed by Bourdon and Pajot [5, 4]. In their examples $\partial I_{p,q}$ is homeomorphic to the Menger curve, and it turns out to be possible to precisely compute the conformal dimension of $\partial I_{p,q}$. Note that, similar to the Sierpinski carpet, the Menger curve is not topologically co-Hopf.

Problem 1.12. Are the Burdon-Pajot buildings $I_{p,q}$ quasi-isometrically co-Hopf? Equivalently, are their boundaries $\partial I_{p,q}$ quasi-symmetrically co-Hopf?

It is also interesting to investigate quasi-isometric and coarse co-Hopfity for other natural classes of groups and metric spaces. In an ongoing work (in preparation), Jason Behrstock, Alessandro Sisto, and Harold Sultan study quasi-isometric co-Hopfity for mapping class groups and also characterize exactly when this property holds for fundamental groups of 3-manifolds.

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2. GEOMETRIC OBJECTS

2.1. Horoballs. Recall that, by Convention 1.8, X is a rank one symmetric space different from $\mathbb{H}_{\mathbb{R}}^2$. Namely, X is isometric to a hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$ (with $n \geq 3$), $\mathbb{H}_{\mathbb{C}}^n$ (with $n \geq 2$), \mathbb{H}_H^n over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}_{\mathbb{O}}^2$. We recall some properties of X . See [6], Chapter II.10, for details.

Definition 2.1. Let $0 \in X$ be a basepoint and γ a geodesic ray starting at 0. The associated function $b : X \rightarrow \mathbb{R}$ given by

$$(2.1) \quad b(x) = \lim_{s \rightarrow \infty} d(x, \gamma(s)) - s$$

is known as a Busemann function on X . A *horosphere* is a level set of a Busemann function. The set $b^{-1}[t_0, \infty) \subset X$ is a *horoball*. Up to the action of the isometry group on X , there is a unique Busemann function, horosphere, and horoball.

A Busemann function $b(x)$ provides a decomposition of X into *horospherical coordinates*, a generalization of the upper-halfspace model. Namely, let $\sigma = b^{-1}(0)$ and decompose $X = \sigma \times \mathbb{R}^+$ as follows: given $x \in X$, flow along the gradient of b for time $b(x)$ to reach a point $s \in \sigma$, and write $x = (s, e^{b(x)})$. In horospherical coordinates, the σ -fibers $\{s\} \times \mathbb{R}^+$ are geodesics, the \mathbb{R}^+ -fibers $\sigma \times \{t_0\}$ are horospheres, and the sets $\sigma \times [t_0, \infty)$ are horoballs. Other horoballs appear as closed balls tangent to the boundary $\sigma \times \{0\}$.

If (M, d) is a metric space and $C \geq 0$, a path $\gamma : [a, b] \rightarrow M$, parameterized by arc-length, is called a *C-rough geodesic* in M , if for any $t_1, t_2 \in [a, b]$ we have

$$(2.2) \quad |d(\gamma(t_1), \gamma(t_2)) - |t_1 - t_2|| \leq C.$$

If Y, Y' are metric spaces, a map $f : Y \rightarrow Y'$ is *coarsely Lipschitz* if there exists $C > 0$ such that for any $y_1, y_2 \in Y$ we have $d_{Y'}(f(y_1), f(y_2)) \leq C d_Y(y_1, y_2)$. If Y is a path metric space then it is easy to see that $f : Y \rightarrow Y'$ is coarsely Lipschitz if and only if there exist constants $C, C' > 0$ such that for any $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) \leq C$ we have $d_{Y'}(f(y_1), f(y_2)) \leq C'$.

The following two lemmas appear to be well known folklore facts:

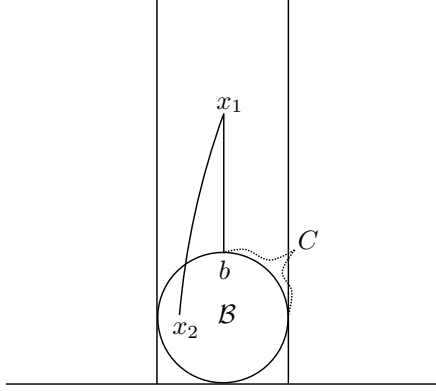
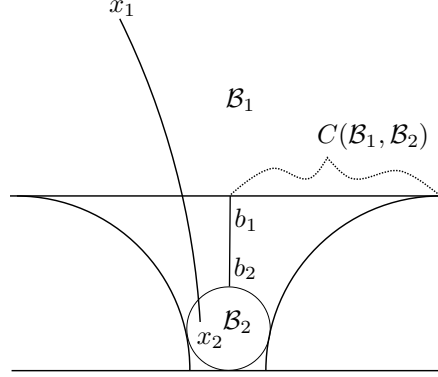
Lemma 2.2. *There exists $C > 0$ with the following property: Let \mathcal{B} be a horoball in X , $x_1 \in X \setminus \mathcal{B}$ and $x_2 \in \mathcal{B}$. Let b be the point in \mathcal{B} closest to x_1 . Then the piecewise geodesic $[x_1, b] \cup [b, x_2]$ is a C -rough geodesic.*

Proof. Acting by isometries of X , we may assume that \mathcal{B} is a fixed horoball that is tangent to the boundary of X in the horospherical model. We may also assume that b is the top-most point of \mathcal{B} , so that x_1 lies in the vertical geodesic passing through b . See Figure 1.

Consider the “top” of \mathcal{B} , i.e. the maximal subset of $\partial\mathcal{B}$ that is a graph in horospherical coordinates. Considering the Riemannian metric on X in horospherical coordinates, one sees that the geodesic $[x_1, x_2]$ must pass through the top of \mathcal{B} . Setting C to be the radius of the top of \mathcal{B} , centered at b , completes the proof. \square

Lemma 2.3. *Let $\mathcal{B}_1, \mathcal{B}_2$ be disjoint horoballs, and $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2$. Let $[b_1, b_2]$ be the minimal geodesic between \mathcal{B}_1 and \mathcal{B}_2 . Then $[x_1, b_1] \cup [b_1, b_2] \cup [b_2, x_2]$ is a C -rough geodesic, for the value of C in Lemma 2.2.*

Proof. The proof is analagous to that of Lemma 2.2. We may normalize the horoballs $\mathcal{B}_1, \mathcal{B}_2$ as in Figure 2. The normalization depends only on the distance

FIGURE 1. Lemma 2.2
for $X = \mathbb{H}_{\mathbb{R}}^2$.FIGURE 2. Lemma 2.3
for $X = \mathbb{H}_{\mathbb{R}}^2$.

$d(\mathcal{B}_1, \mathcal{B}_2)$. Any geodesic $[x_1, x_2]$ must then pass through compact regions near b_1 and b_2 . Let $C(\mathcal{B}_1, \mathcal{B}_2)$ be the radius of this region in \mathcal{B}_1 . Fixing \mathcal{B}_1 and varying \mathcal{B}_2 , set $C = \sup C(\mathcal{B}_1, \mathcal{B}_2)$. The value $C(\mathcal{B}_1, \mathcal{B}_2)$ remains bounded if the distance between the horoballs goes to infinity (converging to the constant C in Lemma 2.2). Thus, the infimum is attained and $C < \infty$. This completes the proof. \square

Lemma 2.4. *Let $\mathcal{B}_1, \mathcal{B}_2$ be disjoint horoballs, $x_1 \in \mathcal{B}_1$, $x_2 \in \mathcal{B}_2$. Denote the minimal geodesic between \mathcal{B}_1 and \mathcal{B}_2 by $[b_1, b_2]$. Then $d(x_1, b_1) \leq d(x_1, x_2)$.*

Proof. Fix $D > 0$ and allow $\mathcal{B}_1, \mathcal{B}_2, x_1 \in \mathcal{B}_1$, and $x_2 \in \mathcal{B}_2$ to vary with the restriction $d(x_1, x_2) = D$. Define a function f on the interval $[0, D]$ by

$$f(t) = \sup\{d(x_1, b_1) : d(\mathcal{B}_1, \mathcal{B}_2) = t\},$$

where the supremum is over all combinations of the variables with the restriction stated above, and b_1 denotes the closest point of \mathcal{B}_1 to \mathcal{B}_2 . Then f is a decreasing function, since increasing t pushes the horoballs farther apart and forces x_1 closer to x_2 . In particular, $f(D) = 0$ since necessarily $x_1 = b_1$. Conversely, $f(0) = D$, taking $x_2 = b_1 = b_2$. We then have for any choice of disjoint $\mathcal{B}_1, \mathcal{B}_2$ and x_1, x_2 in the corresponding horoballs, that

$$d(x_1, b_1) \leq f(d(\mathcal{B}_1, \mathcal{B}_2)) \leq d(x_1, x_2) = D,$$

as desired. \square

2.2. Truncated Spaces.

Definition 2.5. Let $X \neq \mathbb{H}_{\mathbb{R}}^2$ be a negatively curved rank one symmetric space. A *truncated space* Ω is the complement in X of a set of disjoint open horoballs. A truncated space is *equivariant* if there is a (non-uniform) lattice $\Gamma \subset \text{Isom}(X)$ that leaves Ω invariant, with Ω/Γ compact.

We will consider Ω with the induced path metric d_Ω from X . Under this metric, curvature remains negative in the interior of Ω . The curvature on the boundary need not be negative. For an extensive treatment of truncated spaces, see [16].

Remark 2.6. Note that truncated spaces are, in general, not uniquely geodesic. Specifically, if X is not a real hyperbolic space, then components of $\partial\Omega$ (which come from horospheres in X) are isometrically embedded in (Ω, d_Ω) copies of non-uniquely-geodesic Riemannian metrics on certain nilpotent groups. In particular, (Ω, d_Ω) is not necessarily a $CAT(0)$ -space.

Remark 2.7. Let X be a negatively curved rank one symmetric space and $\Gamma \subset \text{Isom}(X)$ a non-uniform lattice. Then X/Γ is a finite-volume manifold with cusps. In X , each cusp corresponds to a Γ -invariant family of horoballs. Removing the horoballs produces an equivariant truncated space Ω whose quotient Ω/Γ is the compact core of X/Γ .

Proposition 2.8. *Let X be a negatively curved rank one symmetric space and $\Omega \subset X$ an equivariant truncated space. Then the inclusion $\iota : (\Omega, d_\Omega) \hookrightarrow (X, d_X)$ is a coarse embedding.*

Proof. Since d_Ω and d_X are path metrics with the same line element, we have

$$(2.3) \quad d_X(x, y) \leq d_\Omega(x, y)$$

To get the lower bound, define an auxilliary function

$$(2.4) \quad \beta(s) = \max \{d_\Omega(x, y) : x, y \in \Omega \text{ and } d_X(x, y) \leq s\}.$$

Let K be a compact fundamental region for the action of Γ on Ω . Because Γ acts on Ω by isometries with respect to both metrics d_X and d_Ω , we may equivalently define $\beta(s)$ by

$$(2.5) \quad \beta(s) = \max \{d_\Omega(x, y) : x \in K, y \in \Omega \text{ and } d_X(x, y) \leq s\}.$$

Because K is compact and the metrics d_X, d_Ω are complete, $\beta(s) \in (0, \infty)$ for $s \in (0, \infty)$. Furthermore, $\beta : [0, \infty] \rightarrow [0, \infty]$ is continuous and increasing, with $\beta(0) = 0$. Because horospheres have infinite diameter for both d_X and d_Ω (they are isometric to appropriate nilpotent Lie groups with left-invariant Riemannian metrics, see [16]), we also have $\beta(\infty) = \infty$.

Let β' be an increasing homeomorphism of $[0, \infty]$ with $\beta'(s) \geq \beta(s)$ for all s and consider its inverse $\alpha(t)$. For $x, y \in \Omega$ we then have

$$\begin{aligned} d_\Omega(x, y) &\leq \beta(d_X(x, y)) \leq \beta'(d_X(x, y)), \\ \alpha(d_\Omega(x, y)) &\leq d_X(x, y). \end{aligned}$$

This concludes the proof. \square

Remark 2.9. A more precise quantitative version of Proposition 2.8 can be obtained by studying geodesics in Ω , see [10].

2.3. Mappings between truncated spaces. For this section, let $\Omega \subset X$ be a truncated space, with $X \neq \mathbb{H}_{\mathbb{R}}^2$, and $f : \Omega \rightarrow \Omega'$ a d_Ω -quasi-isometric embedding. To ease the exposition, we refer to the target truncated space as $\Omega' \subset X'$.

Lemma 2.10 (Schwartz [16]). *There exists $C > 0$ so that for every boundary horosphere σ of Ω , there exists a boundary horosphere σ' of Ω' such that $f(\sigma)$ is contained in a C -neighborhood of σ' .*

Using nearest-point projection, we may assume $f(\sigma) \subset \sigma'$.

Definition 2.11. Let $\mathcal{B}, \mathcal{B}'$ be horoballs with boundaries σ, σ' . A point in σ corresponds, in horospherical coordinates, to a geodesic ray in B . A map $\sigma \rightarrow \sigma'$ then extends to a map $\mathcal{B} \rightarrow \mathcal{B}'$ in the obvious fashion.

In view of Lemma 2.10, a d_Ω -quasi-isometric embedding $f : \Omega \rightarrow \Omega'$ likewise extends to a map $f : X \rightarrow X'$ by filling the map on each boundary horoball.

Lemma 2.12 (Schwartz [16]). *A quasi-isometry $f : \sigma \rightarrow \sigma'$ induces a quasi-isometry $\mathcal{B} \rightarrow \mathcal{B}'$, with uniform control on constants.*

Idea of proof. One considers the metric on the horospheres of \mathcal{B} parallel to σ , or alternately fixes a model horosphere and varies the metric. One then shows that if f is a quasi-isometry with respect to one of the horospheres, it is also a quasi-isometry with respect to the horospheres at other horo-heights. One then decomposes the metric on \mathcal{B} into a sum of the horosphere metric and the standard metric on \mathbb{R} , in horospherical coordinates. This replacement is coarsely Lipschitz, so the extended map is also coarsely Lipschitz. Taking the inverse of f completes the proof. \square

3. COMPACTLY SUPPORTED COHOMOLOGY

Definition 3.1. Let X be a simplicial complex and $K_i \subset X$ nested compacts with $\cup_i K_i = X$. Compactly supported cohomology $H_c^*(X)$ is defined by

$$(3.1) \quad H_c^*(X) = \varinjlim H^*(X, X \setminus K_i).$$

For a compact space X , $H_c^*(X) = H^*(X)$ but the two do not generally agree for unbounded spaces. We have $H_c^n(\mathbb{R}^n) = \mathbb{Z}$ and $H_c^n(\overline{\Omega}) = 0$ for a non-trivial truncated space Ω . In fact, one has the following lemma.

Lemma 3.2. *Let $Z \subset \mathbb{R}^n$ be a closed subset. Then $H_c^n(Z) \neq 0$ if and only if $Z = \mathbb{R}^n$.*

Proof. It is well-known that the choice of nested compact sets does not affect $H_c^n(Z)$. Choose the sequence $K_i = \overline{B(0, i)} \cap Z$, the intersection of a closed ball and Z . With respect to the subset topology of Z , the boundary of K_i is given by $\partial_Z K_i := \partial K_i \cap \partial B(0, i)$. We have by excision

$$H^n(Z, K_i) = H^n(K_i, \partial_Z K_i) = \tilde{H}^n(K_i / \partial_Z K_i).$$

Note that $K_i \subset \overline{B(0, i)}$ and $\partial_Z K_i \subset \partial \overline{B(0, i)}$, so $K_i / \partial_Z K_i \subset \overline{B(0, i)} / \partial \overline{B(0, i)}$. Thus, if $K_i \neq B(0, i)$, then $K_i / \partial_Z K_i \subset S^n \setminus \{*\}$. That is, $K_i / \partial_Z K_i$ is a compact set in \mathbb{R}^n , and $\tilde{H}^n(K_i / \partial_Z K_i) = 0$. Thus, if $Z = \mathbb{R}^n$, we have $H_c^n(Z) = \mathbb{Z}$. Otherwise, $H_c^n(Z) = 0$. \square

Compactly supported cohomology is not invariant under quasi-isometries or uniform embeddings. The remainder of this section is distilled from [14], where compactly supported cohomology is generalized to a theory invariant under uniform embeddings. For our purposes, the basic ideas of this theory, made explicit below, are sufficient.

Definition 3.3. Let X be a simplicial complex with the standard metric assigning each edge length 1. Recall that a *chain* in X is a formal linear combination of simplices. The *support* of a chain is the union of the simplices that have non-zero coefficients in the chain. The *diameter* of a chain is the diameter of its support.

An acyclic metric simplicial complex X is *k-uniformly acyclic* if there exists a function α such that any closed chain with diameter d is the boundary of a $k+1$ -chain of diameter at most $\alpha(d)$. If X is *k-uniformly acyclic* for all k , we say that it is *uniformly acyclic*.

Likewise, we say that a metric simplicial complex X is *k-uniformly contractible* if there exists a function α such that every continuous map $S^k \rightarrow X$ with image having diameter d extends to a map $B^{k+1} \rightarrow X$ with diameter at most $\alpha(d)$. If X is *k-uniformly contractible* for all k , we say it is *uniformly contractible*.

Remark 3.4. Rank one symmetric spaces and nilpotent Lie groups (with left-invariant Riemannian metrics) are uniformly contractible and uniformly acyclic.

Lemma 3.5. *Let X, Y be uniformly contractible and geometrically finite metric simplicial complexes and $f : X \rightarrow Y$ a uniform embedding. Then there exists an iterated barycentric subdivision of X and $R > 0$ depending only on the uniformity constants of f, X , and Y such that f is approximated by a continuous simplicial map with additive error of at most R .*

Proof. We first approximate f by a continuous (but not simplicial) map by working on the skeleta of X . Starting with the 0-skeleton, adjust the image of each vertex by distance at most 1 so that the image of each vertex of X is a vertex of Y . Next, assuming inductively that f is continuous on each k -simplex of X , we now extend to the $k+1$ skeleton using the uniform contractibility of Y . Since error was bounded on the k -simplices, it remains bounded on the $k+1$ -skeleton.

Now that f has been approximated by a continuous map, a standard simplicial approximation theorem replaces f by a continuous simplicial map, with bounded error depending only on the geometry of X and Y (see for example the proof of Theorem 2C.1 of [13]). \square

Lemma 3.6. *Let X and Y be uniformly acyclic simplicial complexes and $f : X \rightarrow Y$ a uniform embedding. Suppose furthermore that f is a continuous simplicial map. Then if $H_c^n(X) \cong H_c^n(fX)$.*

Proof. We first construct a left inverse ρ of the map $f_* : C_*(X) \rightarrow C_*(fX)$ induced by f on the chain complex of X , up to a chain homotopy P . That is, P will be a map $C_*(X) \rightarrow C_{*+1}(X)$ satisfying, for each $c \in C_*(X)$, the homotopy condition

$$(3.2) \quad \partial P c = c - \rho f_* c - P \partial c$$

and furthermore with diameter of Pc controlled uniformly by the diameter of c .

We start with the 0-skeleton. Each vertex $v' \in fX$ is the image of some vertex $v \in X$ (not necessarily unique). Set $\rho(v') = v$, and extend by linearity to $\rho : C_0(fX) \rightarrow C_0(X)$. To define P , let v be an arbitrary vertex in X and note that $\partial v = 0$. We have to satisfy $\partial P v = v - \rho f_* v$. Since X is acyclic, there exists a 1-chain Pv satisfying this condition. Furthermore, note that $\rho f_* v$ is, by construction, a vertex such that $f(\rho f_* v) = f(v)$. Since f is a uniform embedding, $d(\rho f_* v, v)$ is uniformly bounded above. Thus, Pv may be chosen using *uniform acyclicity* so that its diameter is also uniformly bounded above.

Assume next that ρ and P are defined for all $i < k$ with uniform control on diameters. Let σ be a k -simplex in X . Then $\partial \rho f_* \sigma$ is a chain in X whose diameter is bounded independently of σ . Then, by uniform acyclicity there is a chain σ' with

$\partial\sigma' = \partial\rho f_*\sigma$. We define $\rho(\sigma) = \sigma'$. As before, we need to link σ' back to σ . We have

$$(3.3) \quad \partial(\sigma - \sigma' - P\partial\sigma) = \partial\sigma - \rho f_*\partial\sigma - \partial P\partial\sigma.$$

By the homotopy condition 3.2, we further have

$$(3.4) \quad \partial(\sigma - \sigma' - P\partial\sigma) = \partial\sigma - \rho f_*\partial\sigma - (\partial\sigma - \rho f_*\partial\sigma - P\partial\partial\sigma) = 0.$$

Thus, by bounded acyclicity there is a $k+1$ chain $P\sigma$ such that

$$(3.5) \quad \partial P\sigma = \sigma - \sigma' - P\partial\sigma,$$

as desired. We extend both ρ and P by linearity to all of $C_k(fX)$ and $C_k(X)$, respectively.

To conclude the argument, let K be a compact subcomplex of X and consider the complex $X/(X \setminus K) = K/\partial K$. The maps P and $\rho \circ f_*$ on $C_*(X)$ induce maps on $C_*(K/\partial K)$, and the condition $\partial Pc + P\partial c = c - \rho f_*c$ remains true for the induced maps and chains.

Because chain-homotopic maps on C_* induce the same maps on homology, we have, for $h \in H_*(K/\partial K)$, $h = \rho f_*h$. Conversely, $f_*\rho$ is the identity on cell complexes, so still the identity on homology. Thus, $H_*(K/\partial K) \cong H_*(fK/\partial fK)$. By duality, $H^*(fK/\partial fK) \cong H^*(K/\partial K)$.

Taking K_i to be an exhaustion of X by compact subcomplexes and taking a direct limit, we conclude that $H_c^*(X) \cong H_c^*(fX)$. \square

Corollary 3.7. *Let X and Y be uniformly acyclic simplicial complexes and $f : X \rightarrow Y$ a uniform embedding. There exists an $R > 0$ depending only on the uniformity constants of f, X , and Y so that $H_c^n(N_R(fX)) \cong H_c^n(X)$.*

Proof. Lemma 3.5 approximates f by a continuous simplicial map, within uniform additive error. Lemma 3.6 shows that the resulting approximation induces an isomorphism on compactly supported cohomology. \square

Theorem 3.8 (Coarse co-Hopfity). *Let (X, d_X) be a manifold homeomorphic to \mathbb{R}^n , with d_X a path metric that is uniformly acyclic and uniformly contractible. For each pair of non-decreasing functions $\alpha, \omega : [0, \infty) \rightarrow \mathbb{R}$ with $\alpha(t) < \omega(t)$ and $\lim_{t \rightarrow \infty} \alpha(t) = \infty$, there exists a C' such that any (α, ω) -coarse embedding $f : X \rightarrow X$ is C' -coarsely surjective.*

Proof. By Corollary 3.7, there is a uniform $R > 0$ such that $H_c^n(N_R(fX)) \cong H_c^n(X) \cong \mathbb{Z}$. By Lemma 3.2, $N_R(fX) = X$. Taking $C' = C + 2R$ completes the proof. \square

4. MAIN RESULT

Theorem 4.1 (Quasi-Isometric co-Hopfity). *Let $\Omega \subset X$ and $\Omega' \subset X'$ be equivariant truncated spaces and $f : (\Omega, d_\Omega) \rightarrow (\Omega', d_{\Omega'})$ a quasi-isometric embedding. Then f is coarsely surjective with respect to the truncated metric d_Ω .*

Proof. By Lemma 2.10, we may assume that f maps boundary horospheres of Ω to boundary horospheres of Ω' . By Theorem 3.8, f is a surjection up to a constant independent of the boundary horosphere in question. We then have an extension $F : X \rightarrow X'$, as in Definition 2.11.

By Lemma 2.12, for each boundary horoball \mathcal{B} , the restriction $F|_{\mathcal{B}}$ is a quasi-isometry. By assumption, $F|_{\Omega}$ is a d_{Ω} -quasi-isometry, so $F|_{\Omega}$ is a d -uniform embedding by Proposition 2.8. Since X is a path metric space, F is then coarsely Lipschitz on all of X .

We now show that F is a uniform embedding by establishing a lower bound for distances between image points. Recall that all distances are measured with respect to $d = d_X$ unless another metric is explicitly mentioned.

Let $L \gg 2$ so that F is coarsely L -Lipschitz and $F|_{\mathcal{B}}$ is coarsely L -co-Lipschitz for every boundary horoball \mathcal{B} . Let α, ω be increasing proper functions so that f is an (α, ω) -uniform embedding.

Let $x_1, x_2 \in X$ with $d(x_1, x_2) \gg 0$. We need to provide a lower bound for $d(Fx_1, Fx_2)$ in terms of $d(x_1, x_2)$. Clearly, the lower bound will go to ∞ since F is an isometry along vertical geodesics in horoballs. There are four cases to consider; in all cases we can ignore additive noise by working with sufficiently large $d(x_1, x_2)$ and slightly increasing L .

- (1) Let $x_1, x_2 \in \mathcal{B}$ for the same horoball \mathcal{B} . Then $d(fx_1, fx_2) > d(x_1, x_2)/L$.
- (2) Let $x_1, x_2 \in \Omega$. This case is controlled by the uniform embeddings $\Omega \hookrightarrow X$ and $\Omega' \hookrightarrow X'$ (Proposition 2.8) and the d_{Ω} -quasiisometry constants of f .
- (3) Let $x_1 \in \Omega, x_2 \in \mathcal{B}$ for a horoball \mathcal{B} . Let $b \in B$ be the closest point to x_1 . Then by Lemma 2.2, $[x_1, b] \cup [b, x_2]$ is a C -quasi-geodesic for a universal C depending only on X and X' (see also Figure 1). We consider two sub-cases:
Suppose that $d(x_1, b) > d(x_1, x_2)/L^3$. Let $b' \in f\mathcal{B}$ be the closest point to fx_1 . Then by definition of b , we have

$$d(f^{-1}b', x_1) \geq d(b, x_1) \geq d(x_1, x_2)/L^3.$$

Using Lemma 2.4, we conclude

$$d(fx_1, fx_2) \geq d(b', fx_2) \geq \alpha(d(f^{-1}b', x_2)) \geq \alpha(d(x_1, x_2)/L^3).$$

Suppose, instead, that $d(x_1, b) \leq d(x_1, x_2)/L^3$. Then we have the estimate $d(fx_1, fb) \leq d(x_1, x_2)/L^2$. We also have $d(x_2, b) \approx d(x_1, x_2)$, so $d(fx_2, fb) \geq d(x_1, x_2)/L$. Consider now $b' \in \mathcal{B}$, the closest point to fx_1 . By Lemma 2.2, $d(fb, fb') \leq d(fb, fx_1)$. Thus,

$$d(fx_1, fx_2) \geq d(x_1, x_2)/L - d(x_1, x_2)/L^2.$$

- (4) Let $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2$ be in disjoint horoballs. This case is identical to the previous one, except one uses Lemma 2.3 rather than 2.2.

We have then provided a lower bound for $d(Fx_1, Fx_2)$ for any pair of points $x_1, x_2 \in X$. Thus, the extended map F is a coarse embedding. By Theorem 3.8, F is then coarsely surjective. Namely, there exists $R > 0$ so that $N_R(F(X)) = X'$ (the neighborhood is taken with respect to d).

We now show that the coarse surjectivity of F with respect to d implies the coarse surjectivity of f with respect to d_{Ω} .

Let $\omega' \in \Omega'$ be an arbitrary point. Since F is coarsely surjective, there exists $x \in X$ so that $d_{X'}(f(x), \omega') \leq R$. If $x \in \Omega$, then we have shown that $\omega' \in N_R(f(\Omega))$. Otherwise, x is contained in a horoball associated with Ω . In appropriate horospherical coordinates, the horoball is given by $S \times (t_0, \infty)$ and x can be written as (s_1, t_1) , with $t_1 > t_0$. Likewise, $f(x)$ has coordinates (s'_1, t'_1) , with $(t'_1 > t'_0)$. Furthermore, we have $f(s_1, t_0) = (s'_1, t_0)$. Now, $\omega' \in \Omega'$, so it has horospherical coordinates

(s'_2, t'_2) with $t'_2 < t'_0$. It is easy to see that

$$(4.1) \quad \begin{aligned} R &\geq d_{X'}(\omega', (s'_1, t'_1)) \geq d_{X'}(\omega', (s'_1, t'_0)) \\ &= d_{X'}(\omega', f(s_1, t_0)) \geq d_{X'}(\omega', f(\Omega)). \end{aligned}$$

Thus, for an arbitrary $\omega' \in \Omega'$ we have $d_{X'}(\omega', f(\Omega)) \leq R$. Because $\Omega' \hookrightarrow X'$ is a uniform embedding, this implies that $f : \Omega \rightarrow \Omega'$ is coarsely surjective. \square

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